Martingale Pricing Basics for Traders

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Certain stochastic processes can be classified as Martingale. In simple terms a Martingale can be understood as a stochastic process where expected value of the next step (given all previous steps) is equal to its present value i.e.

$$E\left[X_{t+1}|X_t, X_{t-1}, \ldots X_0\right] = X_t$$

In a more general sense, it follows from above (using tower property of expectation):

$$E\left[X_{t+1}|X_s, X_{s-1}, \ldots X_0\right] = X_s, \forall s \leq t$$

We want to understand martingale processes because we can then use the above property to price derivatives i.e. if an option price process is a martingale its value X_s at $t = s$ can be computed as expected value of its payoff at expiry. However, before going to answer the question of option price being a martingale, we ask another question, is the stock price process defined below using GBM a martingale?

$$\frac{dS}{S} = \mu dt + \sigma dB$$

It seems non-intuitive that stock price should be a martingale. Although in a short span one doesn't expect stock price to change much and the expected value at next instant to remain same as the current value but in the long run stocks should grow, otherwise no one will invest in them. This growth rate above the risk-free rate is the equity risk premia i.e. the risk one takes to accept the variance in prices.

Can we do some kind of transformations in the GBM process to make the stock price a martingale. If drift (μ) is the cause of concern, can we not just drop it, but then we don't model the empirical behaviour of the prices. The way to overcome this dilemma is through the path of more abstract maths of measure spaces.

Change of Measure

Simply put a probability measure can be thought of as assignment of probabilites to different elements of the sample space Ω, such that they add up to 1. For instance, when we roll a dice such that each face has probability $\frac{1}{6}$ of turning up, we can say that in this measure $P(X = i) = \frac{1}{6}, i \in \{1, 2, 3, 4, 5, 6\}$. We can think of another assignments of these probabilities such that sum of probability assignment is 1, and that will be another valid probability measure. When we change the measure, we have gone from a fair dice to a loaded dice. However, the value of random variable remains the same, i.e. it takes values from the same set, just the probability of these values are reassigned.

In terms of a stock price process, we may think of changing measure as assigning different probability to different stock price path. As we discussed for dice, this new probability assignment may be very different from the actual physical reality (physical measure) but this new imaginary probability measure could be simpler to model. If the stock price process in this transformed measure turns out to be a martingale that would be great. Indeed, if we have some tool to go from one measure to another, we can price derivatives in this imaginary world and then transform them back to the physical world.

In measure theory there is a tool known as Radon-Nikodym derivative which helps transform between two probability measures given these probability measures are equivalent. Two measures are called equivalent when all events assigned non-zero probability in one space are also assigned non-zero probability in the other space and vice-versa.

Consider two equivalent measures P and $\tilde{P}$, we can convert all points w in the sample space from one measure to another by using a change of measure function $z(w)$ (called Radon-Nikodym derivative). Consider an event A, we define the probability of event as

$$P(A) = \int_{w \in A} dp(w)$$

We can assign new probabilities to all possible events under the new measure $\tilde{P}$ as

$$\tilde{P}(A) = \int_{w \in A} z(w) dp(w)$$

The function z is basically redistributing probabilities across points in the sample space.

Transforming to Standard Normal

Consider RV $X \sim N(0,1)$ in measure P and $Y = X + \theta$, where θ is a constant. Under measure P , $Y \sim N(\theta, 1)$. Does there exist a measure $\tilde{P}$ in which $Y \sim N(0,1)$? We will now see that we can indeed find such measure. Note the random variable Y remains the same, we just map all points in the space from one measure to another using $z(w)$. We claim that by choosing $z(w) = exp\{-\theta X - \frac{1}{2}\theta^2\}$, the equivalent measure has $Y \sim N(0,1)$

$$P(X \in [a,b]) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$$

Under measure $\tilde{P}$,

$$\tilde{P}(X \in [a,b]) = \int_a^b z(x) \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$$

$$= \int_a^b e^{\{-\theta X - \frac{1}{2}\theta^2\}} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx \quad (1)$$

$$= \int_a^b \frac{1}{\sqrt{2\pi}} e^{\frac{-(x+\theta)^2}{2}} dx \quad (2)$$

Replacing $y = x + \theta$, we get the result that $Y \sim N(0, 1)$ in $\tilde{P}$

Extension to Browninan Motion

This can be further extended to Brownian motion. If $B(t)$ is Brownian in the physical measure P i.e. $B(t) \sim N(0, t)$, it can be shown that there exist a measure Q in which $B(t) + \theta(t)$ is also Brownian without drift, where $\theta(t)$ is a non-stochastic function of time. This is the simple form of famous Grisonov's theorem. The transformation function $z(w)$ is very similar to what we have seen above. We define

$$B^Q(t) = B(t) + \frac{\mu - r}{\sigma} t$$

then from above discussion we know under measure Q, $B^Q(t)$ is standard Brownian. If we replace this in the equation of GBM price process, we get the price process in this imaginary world as:

$$\frac{dS}{S} = rdt + \sigma dB^Q \tag{3}$$

Martingale Stock Price Process

We are one step closer to our aim of finding an imaginary world where stock prices can be martingales but equation 3 is still not a martingale because it is not driftless. Yet, this imaginary world is very interesting because instead of drift μ, stocks in this world grow at risk free rate r, that's why this is also known as risk-neutral measure. In this world the discounted stock price Se^{-rt} is drift less.

$$d(Se^{-rt}) = -rSe^{-rt}dt + e^{-rt}dS \tag{4}$$

$$= -rSe^{-rt}dt + e^{-rt}(rSdt + \sigma SdB^Q) \tag{5}$$

$$= Se^{-rt}\sigma dB^Q \tag{6}$$

$$\frac{d(Se^{-rt})}{Se^{-rt}} = \sigma dB^Q \tag{7}$$

Awesome, so discounted stock price is a martingale under risk-neutral measure. The next step for us to note that if the underlying stock process is a martingale, we can also show that its derivatives are also martingales. However, before we go there we will understand two more concepts: no-arbitrage pricing and replication.

Replication and no-arbitrage

A derivative is said to be replicable if we can create a portfolio of stock and cash (primitive securities) such that:

1. Value of portfolio at maturity is equal to the value of derivative under all possible paths

2. Portfolio is self-financing i.e. there is no inflow or outflow of cash from the portfolio.

As the value of the derivative at maturity is equal to value of portfolio and since the portfolio is self-financing, the value of portfolio and derivative should be same at all times irrespective of the path of the stock, unless there is arbitrage in the market. Under the no-arbitrage assumption, the cost of setting up such a portfolio is the value of the derivative. This is also called no-arbitrage pricing. Interestingly and bit non-intuitive for me, note that the replication has nothing to do with the probability measure we choose. Choosing a different measure only changes the assignment of probabilities to different paths but because the measures are equivalent all non-zero probability paths are consistent across measures. This means the value of replicating portfolio should be same irrespective of the measure chosen and as value of the replicating portfolio is same as the value of derivative, valuations in different measures should be the same (as long as there is no-arbitrage condition).

To summarize, if there is replicability, we can setup a portfolio paying same as the derivative at expiry and as no-arbitrage holds, this portfolio is valued same as the derivative at all earlier times. As the above argument has nothing to do with the measure under consideration, pricing under all equivalent measures should be the same. Finally, as pricing under martingale measure is simpler, we choose that over other measures.

Coming back to our original question: if stock process is martingale in Q-measure, does that make derivative process also a martingale. The answer is yes, and it can be understood through the replication argument. Let's say a derivative C is replicated using a portfolio Π of δ_0 cash and δ_1 stock at time $t = 0$.

$$\Pi(0) = \delta_0 + \delta_1 S(0)$$

At next step $t = 1$, sincce this is a replicating portfolio, its value is equal to the value of derivative $C(1)$:

$$\Pi(1) = \delta_0 e^r + \delta_1 S(1) = C(1)$$

Taking e^r on the other side and taking expectation wrt to martingale measure Q:

$$E^Q[\Pi(1)e^{-r}] = \delta_0 + \delta_1 E^Q[S(1)e^{-r}] = E^Q[C(1)e^{-r}] \quad (8)$$

$$E^Q[C(1)e^{-r}] = \delta_0 + \delta_1 S(0) = \Pi(0) \quad (9)$$

As under no-arbitrage $\Pi(0) = C(0)$, we can see that discounted derivative price is also a martingale. Thus, it is enough to find a measure in which stock is martingale and the derivative will follow with replication and no-arbitrage.

Finally we state two well-known theoretical results but without any notations and theoretical rigour. These are the fundamental theorems of asset pricing (FTAP) and they show why we can always do valuation under the equivalent martingale measure (Q) given no-arbitrage and replicability holds.

1. FTAP1: No arbitrage model implies existence of at least one equivalent martingale measure. Existence of at least one equivalent martingale measure implies no-arbitrage.

2. FTAP2:If the market is complete i.e. each and every claim can be replicated by primitive securities and there is no-arbitrage then there exist exactly one equivalent martingale measure. This means the price of every

derivative is unique and that is the cost of replication which is same as discounted expected value of payoff at expiry under risk-neutral measure.

References

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