Stochastic Calculus Basics for Traders

Piyush Bhardwaj

April 13, 2024

1 Stochastic Process

We start by thinking about stock price as a random and time dependent quantity. We model such evolving random phenomena by thinking of them as stochastic (or random) processes. To be able to study them, we first need to define a stochastic process. Without going into too much technicalities, we can start simply with random variables. A random variable (RV) takes value from a set of outcomes (sample space) as a result of some random event, for instance roll of a dice or toss of a coin (in case of stock prices this random event is multiple participants quoting and trading in the market). A stochastic process can be thought of as a sequence of random variables indexed by numbers from an index set. The index set can be discrete or continuous. For modelling stock price $S(t)$ index set is continuous time t.

Once we have decided to model price as stochastic process, next question is what kind of distribution does each RV in this sequence follow. For this, we take reliance on our intuition and some simplification. We note that change in stock price in a given interval can be assumed to be normally distributed with mean 0 and varianceΔt. Also, we may assume that price change in two non-overlapping intervals is independent of each other. One may think that this is not necessarily true, prices may exhibit momentum and the mean $= 0$ assumption is also not true as prices show drift. However, these are simplifying assumptions to build a framework to quantitatively study stock prices. As we build the theory, we will study in later notes some stochastic processes that can take care of these assumptions.

1.1 Brownian Motion

Given the setup above, we now describe a simple stochastic process for modelling stock prices. We assume that the process starts at 0, and at each moment the change in its value is normally distributed and independent of previous changes. This indeed is a known and well studied random process called Brownian motion. We can define it as:

1. $B(0) = 0$
2. $B(t) - B(s) \sim N(0, t - s)$
3. $B(t_i) - B(s_i)$ are independent over non overlapping intervals

1.2 Properties of Browning motion

1.2.1 Non-differentiable

Next we want to model change in $B(t)$ in a small duration Δt. First we state that $B(t)$ is not differentiable i.e. $\lim_{h->0} \frac{B(t+h)-B(t)}{h}$ doesn't exist. This is not intuitive, one would suppose the niceties of the normal distribution to play out and not let the B jump around too much in small duration Δt. Also, when we visualize stock prices they do feel smooth to the naked eyes. However it's indeed true, as $\Delta t \to 0$, change in B could be arbitrarily large and that means we can't draw a line along $B(t)$ and define a slope. The intuitive reason is that the change in function $B(t)$ in a given interval of time, howsoever small, is stochastic and thus can take any arbitrary value. Hence, we really can't think of $B(t)$ as a smooth function.

1.2.2 Quadratic Variation

Consider function $f(x) = x$ and interval $t = 0$ to $t = T$. We split this in tiny N intervals where each interval is of length $\frac{T}{N}$, we then compute this function's quadratic variation as:

$$QV = \sum \{f(t_{i+1}) - f(t_i)\}^2 = \sum (T/N)^2 = T^2/N \tag{1}$$

As $N \to \infty$, $QV \to 0$. This is understandable, as we refine our interval the quadratic terms will be too small and thus the sum is 0. You may also think that for each small interval we can approximate function by its slope and ignore the higher order terms. This holds true for all well behaved functions that we generally encounter. However, this can't be said for $B(t)$. We have seen that B is not differentiable thus the slope is not defined, the quadratic changes at every small interval accumulate. Indeed,

$$\sum (B(t_{i+1}) - B(t_i))^2 = T \tag{2}$$

To see this, $B(t_{i+1}) - B(t_i) \sim N(0, \frac{T}{N})$. QV is the sum of square of iid normal RV. Each has mean $\frac{T}{N}$, thus the sum is T in expectation. It can be shown to hold with probability 1 as well using law of large numbers. So, we see quadratic variation property is $\sum (B(t_{i+1}) - B(t_i))^2 = \sum \Delta B^2 = T$ and we can also write $\Delta B^2 = \Delta t$. I think we can show the last result to be true with probability 1, but I am not sure.

1.3 Ito's Calculus

Classical calculus tools are not available for stochastic process B as the differential operator can't be defined in the classical sense. However, often we want to study small change in a smooth function of a stochastic process $f(B_t)$. In order to understand how we can compute this $\Delta f(B_t)$, we first try to compute this for a regular function $f(x)$ using Taylor's expansion at x:

$$\Delta f(x) = f(x + \Delta x) - f(x) \tag{3}$$

$$= f(x) + f'(x)\Delta x + \frac{1}{2} f''(x)\Delta x^2 - f(x) \tag{4}$$

Ignoring quadratic and above terms for small Δx, we get:

$$\Delta f(x) = f'(x)\Delta x \tag{5}$$

We can use the similar approach to compute $\Delta f(B_t)$:

$$\begin{aligned}
\Delta f(B_t) &= f(B_t + \Delta B_t) - f(B_t) && (6)\\
&= f(B_t) + f'(B_t)\Delta B_t + \frac{1}{2} f''(B_t)\Delta {B_t}^2 - f(B_t) && (7)\\
&= f'(B_t)\Delta B_t + \frac{1}{2} f''(B_t)\Delta t && (8)
\end{aligned}$$

We can now define $\Delta B_t = dB$ for small dt and drop the t notation, this leads us to one of our main result (ito1):

$$df = f'(B)dB + \frac{1}{2} f''(B)dt \tag{9}$$

1.3.1 Ito's Lemma: Extension to $f(t, X)$

For a more generalized two variable function of non-stochastic t and a stochastic process X

$$df(t, X) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial X} dX + \frac{1}{2} \frac{\partial^2 f}{\partial X^2} dX^2 \tag{10}$$

For a process defined simply as $X = B_t$ or $dX = dB$ we can replace $dX^2 = dt$

$$df(t, X) = \left(\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial X^2} \right) dt + \frac{\partial f}{\partial X} dB \tag{11}$$

For another slightly involved stochastic process defined as $dX_t = \mu_t dt + \sigma_t dB$ we can substitute in equation 10 to get:

$$\begin{aligned}
df(t, X) &= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial X} (\mu_t dt + \sigma_t dB) + \frac{1}{2} \frac{\partial^2 f}{\partial X^2} (\mu_t dt + \sigma_t dB)^2 && (12)\\
&= \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial X} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 f}{\partial X^2} \right) dt + \sigma_t \frac{\partial f}{\partial X} dB && (13)
\end{aligned}$$

Here we have ignored higher order dt^2 and $dBdt$ terms.

1.3.2 Extension to Geometric Brownian Motion (GBM)

Further, we extend application of Ito's calculus to GBM which is often used to model stock prices. Under GBM, instantaneous stock returns are modeled with a deterministic drift component μ and a stochastic Brownian component with variance σ^2 :

$$\frac{dX}{X} = \mu dt + \sigma dB$$

Using 10 we note:

$$\begin{aligned}
df(t, X) &= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial X} (\mu X dt + \sigma X dB) + \frac{1}{2} \frac{\partial^2 f}{\partial X^2} (\mu X dt + \sigma X dB)^2 && (14)\\
&= \left(\frac{\partial f}{\partial t} + \mu X \frac{\partial f}{\partial X} + \frac{1}{2} \sigma^2 X^2 \frac{\partial^2 f}{\partial X^2} \right) dt + \sigma X \frac{\partial f}{\partial X} dB && (15)
\end{aligned}$$

1.4 Conclusion

That's it! These are the main results to know in stochastic calculus to be able to understand some of the derivatives pricing equations like BSM. There is endless more maths involved but I think understanding the above should be enough to compute on our own some of the arithmetic. For instance, the classical calculus result of product rule and quotient rule can be extended to ito's calculus as well.

- Classical: $d(XY) = XdY + YdX$
- Ito's: $d(XY) = XdY + YdX + dXdY$

This result can be obtained by writing higher order terms in classical calculus formula of differential of two variable function $f(X, Y)$.

$$df(X,Y) = \frac{\partial f}{\partial Y}dY + \frac{\partial f}{\partial X}dX + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}dX^2 + \frac{1}{2}\frac{\partial^2 f}{\partial Y^2}dY^2 + \frac{\partial^2 f}{\partial X \partial Y}dXdY \quad (16)$$

For $f(X, Y) = XY$ we can get the above result. Similarly, lets try for $f(X, Y) = \frac{X}{Y}$

$$df(X,Y) = -\frac{X}{Y^2}dY + \frac{1}{Y}dX + \frac{X}{Y^3}dY^2 - \frac{1}{Y^2}dXdY \quad (17)$$

1.5 References

- MIT Topics in Mathematics with Applications in Finance
- QuantPie