Option Prices Bounds and Probability Density Function

Piyush Bhardwaj

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From our previous notes we know that an arbitrage-free market is equivalent to having a risk-neutral measure in which discounted option prices behave as martingales. Price of a call option at strike K and at time t, represented by $C(S_t, t, K, T, r, q)$ where r is risk free rate, q is dividend rate and T is maturity can thus be written as :

$$C(S_t, t, K) = E^Q[e^{-r\tau} C(S_T, T, K|S_t, t)] \tag{1}$$

We have dropped r,q and T for simplicity and $\tau = T - t$ Further expanding the expectation as integral we get for European call option price as:

$$C(S_t, t, K) = e^{-r\tau} \int_0^\infty max(S_T - K, 0) p^Q(S_T, T|S_t, t) dS_T \tag{2}$$

where $p^Q(S_T, T|S_t, t)$ is the probability density function of the stock price under the risk-neutral measure. We will drop the Q subscript going forward for simplicity. We can simplify it further by breaking the max terms using limits of integration:

$$C(S_t, t, K) = e^{-r\tau} \int_0^K max(S_T - K, 0) p(S_T, T|S_t, t) dS_T + e^{-r\tau} \int_K^\infty max(S_T - K, 0) p(S_T, T|S_t, t) dS_T \tag{3}$$

$$= e^{-r\tau} \int_K^\infty (S_T - K) p(S_T, T|S_t, t) dS_T \tag{4}$$

We take partial derivative wrt K:

$$\frac{\partial C(S_t, t, K)}{\partial K} = e^{-r\tau} \frac{\partial}{\partial K} \int_K^\infty (S_T - K) p(S_T, T|S_t, t) dS_T \tag{5}$$

$$= -e^{-r\tau} \int_K^\infty p(S_T, T|S_t, t) dS_T \tag{6}$$

The second equivalency follows from application of Leibniz rule. This is very interesting, from this we deduce the following:

1. **Monotonicity** : As $\frac{\partial C(S_t,t,K)}{\partial K} < 0$, calls at higher strike will be priced lower. This makes sense as calls at lower strike will get in the money earlier than calls at higher strike.

2. **Bounds on call spreads**: Price of call spread is always greater than 0 and less than discounted difference between the strikes. This can be noted as $0 < \int_K^\infty p(S_T, T|S_t, t)dS_T < 1$, it follows from equation 5:

$$0 > \frac{C(S_t, t, K_2) - C(S_t, t, K_1)}{K_2 - K_1} > -e^{-r\tau} \tag{7}$$

$$0 < C(S_t, t, K_1) - C(S_t, t, K_2) < -e^{-r\tau}(K_2 - K_1) \tag{8}$$

3. **Convexity**: We will see in section below that $\frac{\partial^2 C(S_t,t,K)}{\partial K^2} > 0$

Risk-Neutral CDF and PDF

Using equation 5 we can evaluate risk-neutral CDF and PDF in form of derivatives of call price. Eq 5 can be written as:

$$\frac{\partial C(S_t, t, K)}{\partial K} = -e^{-r\tau}P(S_T > K|S_t) \tag{9}$$

$$= -e^{-r\tau}(1 - P(S_T < K|S_t)) \tag{10}$$

This shows price of a call spread adjusted by discounted strike difference is the risk-neutral probability of call option ending in the money. Note, $P(S_T < K)$ is the CDF of risk neutral distribution of S_T given S_t. Differentiating again with K therefore gives the risk-neutral probability density of $S_T|S_t$.

$$\frac{\partial^2 C(S_t, t, K)}{\partial K^2} = e^{-r\tau}p(S_T|S_t) > 0 \tag{11}$$

Butterfly price and risk-neutral pdf

Consider a call butterfly position created by buying call at strike K_1 and K_3 and by selling 2 lots of call at strike K_2 i.e. value of portfolio $\Pi(S_t, t; K_1, K_2, K_3)$:

$$\Pi(S_t, t; K_1, K_2, K_3) = C(S_t, t, K_1) + C(S_t, t, K_3) - 2C(S_t, t, K_2) \tag{12}$$

$$= (C(S_t, t, K_1) - C(S_t, t, K_2)) - (C(S_t, t, K_2) - C(S_t, t, K_3)) \tag{13}$$

$$= (K_1 - K_2)\left(\frac{\partial C}{\partial K}|_{K_1} - \frac{\partial C}{\partial K}|_{K_2}\right) \tag{14}$$

$$= (K_1 - K_2)^2 \frac{\partial^2 C}{\partial K^2} \tag{15}$$

In above we have assumend distance between K_1 and K_2 to be same as between K_2 and K_3 for simplicity. This gives us our second bound which is **butterfly prices should always be positive** and price of a butterfly scaled by distance between strikes squared is the pdf .

Calendars

Along expiry dimension only weak results are known. For expires T_1 and T_2, given $T_2 > T_1$, we know for european option of same strike and given 0 dividends

$$C(S_t, t, K, T_2, r, 0) > C(S_t, t, K, T_1, r, 0)$$

where $C(S_t, t, K, T, r, q)$ represents price of a call option at strike K and at time t, where r is risk free rate, q is dividend rate and T is maturity

References

1. Matthias R. Fengler Section 1 and 2 of Arbitrage-free smoothing of the implied volatility surface

2. Chapter 11: Option price and probability Duality An Intraduction to Quantitative Finance by Stephyn Blyth

3. Some of the original results are due to Breeden, D. and Litzenberger, R. (1978). Price of state-contingent claims implicit in options prices, Journal of Business 51: 621–651.